Surface Parameterization

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Outline

- Motivation
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- Angle Preservation
  - Discrete Harmonic Maps
  - Discrete Conformal Maps
  - Angle Based Flattening
- Reducing Area Distortion
- Alternative Domains
Surface Parameterization

Mercator-Projektion

[www.wikipedia.de]

Mollweide-Projektion
Surface Parameterization

[www.wikipedia.de]
Motivation

• Texture mapping

Motivation

- Many operations are simpler on planar domain

Lévy: *Dual Domain Extrapolation*, SIGGRAPH 2003
Motivation

• Exploit regular structure in domain

Gu, Gortler, Hoppe: *Geometry Images*, SIGGRAPH 2002
Surface Parameterization
Surface Parameterization

\[ f(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} \]

\[ J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} \] (Jacobian)
Surface Parameterization

d\mathbf{X} = J \, d\mathbf{U}
Surface Parameterization

\( dX = J \, dU \)

\[ \|dX\|^2 = dU \, J^T J \, dU \]

First Fundamental Form

\[ I = \begin{pmatrix} x_u x_u & x_u x_v \\ x_v x_u & x_v x_v \end{pmatrix} \]
Characterization of Mappings

- By first fundamental form $I$
  - Eigenvalues $\lambda_{1,2}$ of $I$
  - Singular values $\sigma_{1,2}$ of $J$ ($\sigma_i^2 = \lambda_i$)

- **Isometric**
  - $I = \text{Id}$, $\lambda_1 = \lambda_2 = 1$

- **Conformal**
  - $I = \mu \text{Id}$, $\lambda_1 / \lambda_2 = 1$

- **Equiareal**
  - $\det I = 1$, $\lambda_1 \lambda_2 = 1$

angle preserving

area preserving
Piecewise Linear Maps

- Mapping = 2D mesh with same connectivity
Objectives

- Isometric maps are rare
- Minimize distortion w.r.t. a certain measure
  - Validity (bijective map)
  - Boundary
  - Domain
  - Numerical solution

Triangle flip
e.g., spherical
linear / non-linear?
fixed / free?
Discrete Harmonic Maps

- $f$ is \textit{harmonic} if $\Delta f = 0$
- Solve Laplace equation

\[
\begin{align*}
\Delta u &= 0 \\
\Delta v &= 0
\end{align*}
\]

\[(u, v)_{|_{\partial \Omega}} = (u_0, v_0)\]

- In 3D: "fix planar boundary and smooth"
Discrete Harmonic Maps

• $f$ is harmonic if $\Delta f = 0$

• Solve Laplace equation

• Yields linear system (again)

$$L(p_i) = \sum_{j \in N_i} w_{ij} (p_j - p_i) = 0$$

vertices $1 \leq i \leq n$

• Convex combination maps
  - Normalization
  - Positivity

$$\sum_{j \in N_i} w_{ij} = 1$$

$w_{ij} > 0$
Convex Combination Maps

- Every (interior) planar vertex is a \textit{convex combination} of its neighbors
- Guarantees \textit{validity} if boundary is mapped to a convex polygon (e.g., rectangle, circle)
- Weights
  - Uniform \textit{(barycentric mapping)}
  - Shape preserving [Floater 1997]
  - Mean Value Coordinates [Floater 2003]
    - Use mean value property of harmonic functions

Reproduction of planar meshes
Conformal Maps

• Planar conformal mappings $f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ satisfy the Cauchy-Riemann conditions

\[
\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} \quad \text{and} \quad \frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x}
\]
Conformal Maps

• Planar conformal mappings \( f(x, y) = \left( \begin{array}{c} u(x, y) \\ v(x, y) \end{array} \right) \) satisfy the Cauchy-Riemann conditions

\[ u_x = v_y \quad \text{and} \quad u_y = -v_x \]

• Differentiating once more by \( x \) and \( y \) yields

\[ u_{xx} = v_{xy} \quad \text{and} \quad u_{yy} = -v_{xy} \quad \Rightarrow \quad u_{xx} + u_{yy} = \Delta u = 0 \]

\[ \Delta v = 0 \]

and similar

• conformal \( \Rightarrow \) harmonic
Discrete Conformal Maps

- Planar conformal mappings $f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ satisfy the Cauchy-Riemann conditions

  \[ u_x = v_y \quad \text{and} \quad u_y = -v_x \]

- In general, there are no conformal mappings for piecewise linear functions!
Discrete Conformal Maps

- Planar conformal mappings \( f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \) satisfy the Cauchy-Riemann conditions.

\[ u_x = v_y \quad \text{and} \quad u_y = -v_x \]

- Conformal energy (per triangle \( T \))

\[ E_T = (u_x - v_y)^2 + (u_y + v_x)^2 \]

- Minimize

\[ \sum_{T \in \mathcal{T}} E_T A_T \rightarrow \min \]
Discrete Conformal Maps

- Least-squares conformal maps [Lévy et al. 2002]
  \[ \sum_{T \in T} E_T A_T \rightarrow \min \quad \text{where} \quad E_T = (u_x - v_y)^2 + (u_y + v_x)^2 \]

- Satisfy Cauchy-Riemann conditions in least-squares sense

- Leads to solution of linear system

- Alternative formulation leads to same solution…
Discrete Conformal Maps

- Same solution is obtained for

\[ \Delta_S u = 0 \]
\[ \Delta_S v = 0 \]
\[ n \times \nabla u \big|_{\partial \Omega} = c \]
\[ n \times \nabla v \big|_{\partial \Omega} = c \]
\[ (u, v) \big|_{\partial \Omega_0} = (u_0, v_0) \]

cotangent weights
Neumann boundary conditions
+ fixed vertices

*Discrete Conformal Maps* [Desbrun et al. 2002]
Discrete Conformal Maps
Discrete Conformal Maps

- Free boundary depends on choice of *fixed* vertices (>1)

ABF
Angle Based Flattening [Sheffer&de Sturler 2000]

- Perserve angles $\Rightarrow$ specify problem in angles
  - Constraints
    - triangle $\alpha + \beta + \gamma - \pi = 0$
    - Internal vertex $\sum_i \alpha_i - 2\pi = 0$
    - Wheel consistency $\prod_i \sin(\beta_i) - \prod_i \sin(\gamma_i) = 0$
  - Objective function
    \[
    f(x) = \sum_{i=1}^{N} w_i (\alpha_i - \alpha_i^*)^2
    \]
    "optimal" angles (uniform scaling)
Angle Based Flattening

- Free boundary
- Validity: no local self-intersections
- Non-linear optimization
Angle Based Flattening

- Free boundary
- Non-linear optimization
  - Newton iteration
  - Solve linear system in every step

\[
\prod_{i} \sin(\alpha_{i}) - \prod_{i} \sin(\beta_{i}) = 0
\]

\[
\prod_{i} \log \sin(\alpha_{i}) - \prod_{i} \log \sin(\beta_{i}) = 0
\]

[Zayer et al. 2005]
And how about area distortion?
Reducing Area Distortion

- Energy minimization based on
  - MIPS [Hormann & Greiner 2000]
  - modification [Degener et al. 2003]
  - "Stretch" [Sander et al. 2001]
  - modification [Sorkine et al. 2002]
Non-Linear Methods

- Free boundary
- Direct control over distortion
- No convergence guarantees
- May get stuck in local minima
- May not be suitable for large problems
- May need feasible point as initial guess
- May require hierarchical optimization even for moderately sized data sets
Linear Methods

- Efficient solution of a sparse linear system
- Guaranteed convergence
- Fixed convex boundary
- May suffer from area distortion for complex meshes

An alternative approach to reducing area distortion…
  - How accurately can we reproduce a surface on the plane?
  - How do we characterize the mapping?
Reducing Area Distortion

\[ \text{isometry} \quad \left| dX \right| = \left| dx \right| \]
Reducing Area Distortion

- Quasi-harmonic maps [Zayer et al. 2005]

\[ \int C \nabla g \times \nabla g \overset{\text{min}}{\Rightarrow} \]

\[ \text{div}(C \nabla g) = 0 \]

- Iterate (few iterations)
  - Determine tensor $C$ from $f$
  - Solve for $g$
Examples
Examples

\[ \sqrt{\sigma_1 + \sigma_2} \rightarrow \min \]

Stretch metric minimization

Using [Yoshizawa et. al 2004]
Reducing Area Distortion

- Introduce cuts ⇒ area distortion vs. continuity
- Often cuts are unavoidable (e.g., open sphere)

Treatment of boundary is important!
Reducing Area Distortion

- Solve Poisson system [Zayer et al. 2005]

\[ \Delta x = \text{div} \nabla U' \]

* Similar setting used in mesh editing
Spherical Parameterization

• Sphere is natural domain for genus-0 surfaces

• Additional constraint \( \| U \|^2 = 1 \)

• Naïve approach
  - Laplacian smoothing and back-projection
  - Obtain minimum for degenerate configuration
Spherical Parameterization

• (Tangential) Laplacian Smoothing and back-projection
  – Minimum energy is obtained for *degenerate* solution

• Theoretical guarantees are expensive
  – [Gotsman et al. 2003]

• A compromise?!
  – Stereographic projection
  – Smoothing in curvilinear coordinates
Arbitrary Topology

- Piecewise linear domains
  - *Base mesh* obtained by *mesh decimation*
  - Piecewise maps
  - Smoothness
Literature

• Floater & Hormann:  *Surface parameterization: a tutorial and survey*, Springer, 2005


• Desbrun, Meyer, and Alliez: *Intrinsic parameterizations of surface meshes*, Eurographics 2002